GROUP THEORY 2024 - 25, EXERCISE SHEET 5

Exercise 1. *To always do in every course!*
Review the lecture and understand/fill in the gaps in the proofs.

Exercise 2. *Computing some examples of torsion subgroups* (easy)
Determine $\mathrm{Tors}(A)$ for the following examples of abelian groups:

(1) A is a finite abelian group.
(2) $A = (\mathbb{Q}, +)$.
(3) $A = (\mathbb{Q}/\mathbb{Z})$.
(4) $A = \mathbb{C}^\times$.
(5) A is a subgroup of $\mathbb{Z}$.
(6) A is a subgroup of $\mathbb{Z}^k$ for $k \geq 2$.

Exercise 3. (medium) Show that if G is abelian and finitely generated such that $\mathrm{Tors}(G) = G$, then G is a finite group.

Exercise 4. *Free abelian groups* (easy)

Given a family of abelian groups $(A_i)_{i\in I}$ we define their direct sum as the abelian group

$$\bigoplus_{i\in I} A_i = \{(a_i)_{i\in I} | a_i \in A_i, \text{for all } i, \text{only finitely many of the } a_i \text{ are non-zero}\}$$

where the addition is performed component wise.

Prove that the following affirmations are equivalent for an abelian group A:

(1) A is a free abelian group. That is

$$A \cong \mathbb{Z}^{\oplus I} := \bigoplus_{i\in I} \mathbb{Z}.$$

For some indexing set I (not necessarily finite).

(2) There exists a set I and a subset $B = \{a_i | i \in I\} \subset A$ called a **basis**, such that all elements $x \in A$ can be uniquely written as finite sums

$$x = \sum_{k\in I} n_k a_k$$

where all but finitely many n_k equal 0. Note that the uniqueness condition implies that the elements of a basis are linearly independent over $\mathbb{Z}$.

Exercise 5. (medium)
Let $F = \mathbb{Z}^3$ and define a function on a basis (e_1, e_2, e_3) by

$$f(e_1) = (1,0);\ f(e_2) = (1,1),\ f(e_3) = (0,-1).$$

Show that the above assignments define a unique group homomorphism $f : F \to \mathbb{Z}^2$. Is the image of f free abelian ?

Exercise 6. (medium)
Recall that the rank of a finitely generated free abelian group A is the positive integer r such that

$$A \cong \mathbb{Z}^r.$$

Also recall that it was shown in class that if $A \subseteq \mathbb{Z}^r$ is a subgroup, then $A \cong \mathbb{Z}^k$ for some $k \leq r$. We will also see in the next exercise that the rank of a free-abelian group is well defined.

Compute the rank of the following free abelian groups:

(1) Subgroup generated by $(1,1)$ in $\mathbb{Z}^2$.
(2) Subgroup generated by $(1,2)$ and $(-3,-6)$ in $\mathbb{Z}^2$.
(3) $\{a + b\sqrt{2} + c\sqrt{3} \mid a,b,c \in \mathbb{Z}\}$ as an additive subgroup of $\mathbb{R}$.
(4) Subgroup generated by $(2,3,8),(1,5,1)$ and $(1,-9,34)$ in $\mathbb{Z}^3$.
(5) Subgroup generated by $(2,3,8),(1,5,1)$ and $(1,-9,13)$ in $\mathbb{Z}^3$.

Exercise 7. (medium)
Show that the positive rationals $\mathbb{Q}^{>0}$ with group law given by usual multiplication is not a finitely generated abelian group. However show that it is a free-abelian group by exhibiting a basis.

Exercise 8. (medium)
Note that a homomorphism $A \to B$ of abelian groups induces a map

$$\mathrm{Tors}(A) \to \mathrm{Tors}(B).$$

Now let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of abelian groups. Determine if

$$0 \to \mathrm{Tors}(A) \to \mathrm{Tors}(B) \to \mathrm{Tors}(C) \to 0$$

is also exact in general.